A Microfoundation for Normalized CES Production Functions with Factor–Augmenting Technical Change*

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Abstract. We put forward an “endogeneous technology choice” model providing a microfoundation for the aggregate normalized CES production function with non-neutral, factor-augmenting technical change. In this model, firms are allowed to choose unit productivities of capital and labor optimally from a technology menu constructed under the assumption that unit factor productivities (UFP) are independently Weibull-distributed. It is then argued that the Weibull distribution is generically a good approximation of the true UFP distribution if technologies consist of a large number of complementary components. This argument is developed within a novel, tractable, directed R&D setup. The proposed model draws a clear-cut distinction between the direction of R&D and the direction of technical change, and maintains normalization of the production function simultaneously at the local and aggregate level. All our results carry forward to n-input production functions.

Keywords: CES production function, normalization, Weibull distribution, direction of technical change, directed R&D, optimal technology choice

JEL Classification Numbers: E23, E25, O47

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1 Introduction

Aggregate production functions – particularly Cobb–Douglas and CES functions – are used in virtually every paper in contemporary theoretical macroeconomics. Surprisingly few models have been put forward so far, however, in which these functions are derived from microfoundations. The purpose of the current contribution is to enrich this sparse literature with an analytically tractable microeconomic framework able to generate either of these two aggregative specifications endogeneously. It is going to be an idea-based “endogeneous technology choice” model where the aggregate production function is derived as a convex hull of local production functions (LPFs), chosen optimally by homogeneous profit-maximizing firms. Each of these local techniques is in turn characterized by a pair of technology-specific unit factor productivities (UFPs), \((a, b)\), which augment labor and capital, respectively.

Frameworks based on similar premises have been studied recently by Jones (2005) and Growiec (2008a, 2008b).\(^1\) Jones’ (2005) model was founded on the assumption that firms producing the final good sample capital- and labor-augmenting UFPs randomly from a pair of independent Pareto distributions, so that their technology choice is optimal only on average, or in the limit when sufficiently many draws have been made. Growiec (2008a) rewrote Jones’ model into a more tractable form that yields equivalent results and put forward its generalization. The key innovation was to enable the firms to pick their preferred technology pair \((a, b)\) deterministically from a given technology menu, and to shift the stochastic technology-search process to the R&D sector, composed of a continuum of researchers who draw the \((a, b)\) technology pairs from a certain pre-defined joint bivariate distribution. This distribution, in turn, was constructed either from a pair of independent Pareto distributions (mirroring the Jones’ case), or a pair of marginal Pareto distributions dependent according to the Clayton copula (Growiec, 2008a), or a pair of independent Weibull distributions (Growiec, 2008b). In all versions of this

\(^{1}\)The list of related contributions includes also Caselli and Coleman (2006), Nakamura and Nakamura (2008), Nakamura (2009) and León-Ledesma and Satchi (2011), whose “endogeneous technology choice” frameworks are analytically similar to ours. As compared to the current article, however, they are specified in much less detail in terms of the actual microfoundations of the aggregate production function; instead, these articles embed their technology choice frameworks in larger encompassing structures aimed at solving diverse macroeconomic problems. Also, they are endowed with somewhat different interpretations. Finally, Dupuy (2011) provides an alternative microfoundation for the aggregate CES production function, based on a assignment model with heterogeneous workers and tasks.
idea-based model, the shape of the resultant aggregate production function, obtained by plugging the optimal technology choices into LPFs, was found to depend, in general, both on the assumed shapes of LPFs and on the assumed joint distribution of UFPs.

Based on the above assumptions, these three papers have succeeded in providing idea-based microfoundations for Cobb–Douglas and CES aggregate production functions. Jones (2005) has shown that if capital- and labor-augmenting ideas are independently Pareto-distributed, then the aggregate production function is Cobb–Douglas; Growiec (2008a,b) has extended this result by proving that if they are independently Weibull-distributed, or Pareto-distributed and dependent according to the Clayton copula, then the aggregate production function is CES. It has also been demonstrated in these articles that endogeneous technology choice allows the aggregate economy to increase the elasticity of substitution between capital and labor beyond the low level characterizing LPFs.

These papers have overlooked a few important implications of the considered framework, though, most likely because of their somewhat cumbersome parametrization and a number of unnecessary implicit assumptions. The current article identifies several gaps in these papers and fills them, ultimately indicating that the “endogeneous technology choice” model discussed there, once properly parametrized and relieved of unnecessary restrictions, has much more interesting features than it has been uncovered so far.

One of the most striking omissions of these earlier contributions was, in particular, the lack of any theoretical justification for the functional forms of UFP distributions used there, such as, e.g., the Weibull distribution (Growiec, 2008b). The current article fills this gap with a strong argument based on the extreme value property of the Weibull distribution: if each technology is an assembly of a large number of complementary components, then the UFP distributions necessarily become asymptotically Weibull-distributed, for a wide class of distributions from which productivities of these components are drawn. Thanks to this additional argument, the current article provides an even more sound justification for the use of aggregate (normalized) CES production functions in macroeconomics.

Compared to the aforementioned contributions by Jones (2005) and Growiec (2008a,
2008b), the current paper makes four decisive changes in the “endogeneous technology choice” model developed there. Each of them is a source of a distinct contribution to the literature.

First, the model is rewritten in terms of normalized CES functions here (cf. La Grandville, 1989; Klump and La Grandville, 2000). Thanks to this step, we are now able to obtain an interpretable link between the parameters of the microfounded aggregate production function, the LPF, and the bivariate UFP distribution. The reason is that under normalization, with respect to initial inputs $K_0$, $L_0$, output $Y_0$ and the initial capital income share $\pi_0$, parameters of the CES production function represent separate concepts which are otherwise deeply intertwined: e.g., the distribution parameter of the un-normalized CES function is itself a function of the elasticity of substitution and the normalized volume units (cf. Klump and Preissler, 2000). We find that normalization can be maintained simultaneously at the local and at the aggregate level, greatly facilitating the interpretation of the aggregate CES production function’s parameters.

Second, as opposed to the earlier contributions, the current model features also a novel, tractable specification of the R&D sector. On its basis we construct our argument in support for the use of Weibull distributions in the context of R&D productivity (first suggested by Growiec, 2008b, but without any justification). It turns out that if factor-augmenting technologies are inherently complex and consist of a large number of complementary components, then the Weibull distribution should approximate the true productivity distribution better than any anything else, including the celebrated Pareto distribution (see Kortum, 1997; Gabaix, 1999; Jones, 2005, and references therein). The argument is based on the extreme value property of the Weibull distribution (cf. Kotz and Nadarajah, 2000; de Haan and Ferreira, 2006): if one takes $n$ independent draws from some distribution that is bounded from below and satisfies an additional technical assumption (as, e.g., the Pareto, uniform, truncated Gaussian, etc., distributions do) and takes the minimum of these draws, then as $n \to \infty$, this minimum will, after an appropriate normalization, converge in distribution to the standard Weibull distribution.

3This in turn greatly obstructs the estimation of parameters and comparative statics exercises. A thorough elaboration of these issues as well as a survey of the related literature can be found in Klump et al. (2011). These authors also request that the parameters of production functions, derived from microfoundations by Jones (2005) and Growiec (2008a, 2008b), should be provided with an interpretation consistent with normalization. Among other accomplishments, the current paper addresses this request.
with a shape parameter $\alpha > 0$, dependent on the shape of the underlying sampling distribution. Clearly, taking the minimum applies to the case of complex technologies consisting of a number of complementary components (cf. Kremer, 1993; Blanchard and Kremer, 1997; Jones, 2011), because their productivity is then determined by the productivity of their “weakest link”. This complementarity requirement, coupled with the assumption of few substitution possibilities along the LPF, implies that capital and labor should be gross complements along the aggregate CES production function, i.e., that the aggregate (long-run) elasticity of substitution should be less than unity.

Third, the current paper relaxes the assumption implicitly made by Growiec (2008a), that technological progress always augments the technology menu proportionally, as a homothetic transformation from the origin.\footnote{This assumption is also maintained by Jones (2005) and León-Ledesma and Satchi (2011), but due to the multiplicative character of the Cobb–Douglas production function considered there, lifting this restriction does not change any of their results.} In fact, this assumption can be easily generalized, allowing for directed factor-augmenting R&D (cf. Acemoglu, 2003),\footnote{There is a voluminous literature based on Acemoglu’s (2003) framework. Important extensions have been put forward, among others, by Acemoglu (2007) and Acemoglu and Guerrieri (2008).} able to expand the technology menu in selected directions more than in others. This overturns some of the sharp predictions on the direction of technical change, derived by Growiec (2008a). It also helps understand an important theoretical distinction: in endogeneous technology choice models, the direction of R&D (i.e., the direction of expansion of the technology menu) and the direction of technical change, actually observed in an economy, are distinct concepts. The reason is that firms’ technology choices respond not only to the augmentation of the technology menu, but also to factor accumulation.\footnote{In particular, in the neoclassical growth model with a CES aggregate production function and factor-augmenting technical change, these two concepts will be equivalent only along the balanced growth path (BGP) in the unique case when the BGP exists, that is when both R&D and technical change are purely labor-augmenting (cf. Uzawa, 1961). This requires imposing highly specific assumptions on the R&D process. Otherwise, the directions of R&D and technical change will diverge. In the case of a Cobb–Douglas aggregate production function, in turn, the direction of R&D will not have any impact on the direction of technical change: labor-augmenting technical change will always follow changes in output per worker $y$, and capital-augmenting technical change will always reflect changes in output per unit of capital, $y/k$.}

Fourth, the current paper also demonstrates that the considered model is readily generalizable to $n$-factor production functions. Since all the derivations in the $n$-dimensional case are very similar to the two-dimensional case, formal elaboration of this issue has
been delegated to the appendix.

Driven by the logic of our arguments, the structure of the current paper is somewhat unorthodox. Namely, instead of proceeding linearly from assumptions to conclusions, we shall discuss the consecutive layers of our “endogeneous technology choice” framework, starting from the outermost layer and then gradually proceeding inwards. So, in Section 2 we shall derive the aggregate normalized CES production function from idea-based microfoundations, assuming that UFPs of capital and labor are independently Weibull-distributed. In Section 3 we shall demonstrate what would happen if this (static) technology choice model were embedded in a dynamic growth framework, and discuss the model’s implications for the direction of technical change. The following Section 4 should be considered parallel to Sections 2-3, as it deals with the microfoundations of the aggregate Cobb–Douglas production function (cf. Jones, 2005). In particular, it indicates the necessary differences in assumptions with respect to the normalized CES case. In the following Section 5, we shall proceed to the discussion of the details of our directed R&D model, which leads to the functional form of the technology menu faced by firms in Sections 2-3 and provides the generic conditions under which UFPs must be asymptotically Weibull-distributed. Section 6 concludes.

The proposed “endogeneous technology choice” model can be extended in numerous directions, some of which are discussed in the appendices. In Appendix A, we repeat our derivations for the generalized case of $n$-factor production functions. In Appendix B, we deal with the special (empirically implausible but theoretically possible) cases of the model which allow for gross substitutability of inputs along the aggregate CES production function. Appendix C discusses a reinterpretation of selected results in terms of Hicks-neutral technology adoption costs.

2 Microfoundations behind the aggregate normalized CES production function

In the current section, we shall show how one obtains the aggregate normalized CES production function from idea-based microfoundations, i.e., as a convex hull of LPFs, computed under the restriction that UFPs must be chosen from the given technology menu. The assumption regarding the postulated shape of this menu will be subsequently modified in Section 4 (where the aggregate Cobb–Douglas production function
is derived), whereas in Section 5, it will be motivated with a more involved model of
directed R&D and presented as a proposition.

2.1 Framework

The discussed “endogeneous technology choice” framework is based on the following
assumptions.

Assumption 1 The local production function (LPF) takes either the normalized CES
or the normalized Leontief form:

\[ Y = \begin{cases} 
Y_0 \left( \frac{bK}{b_0K_0} \right)^{\frac{\theta}{1-\sigma_{LPF}}} + (1 - \pi_0) \left( \frac{aL}{a_0L_0} \right)^{\frac{\theta}{1-\sigma_{LPF}}} & , \text{if } \sigma_{LPF} \in (0,1), \\
Y_0 \min \left\{ \left( \frac{bK}{b_0K_0} \right), \left( \frac{aL}{a_0L_0} \right) \right\} & , \text{if } \sigma_{LPF} = 0,
\end{cases} \]

where \( \theta \in [-\infty, 0) \) is the substitutability parameter, related to the elasticity of substi-
tution along the LPF via \( \sigma_{LPF} = \frac{1}{1-\theta} \). Leontief LPFs, with \( \sigma_{LPF} = 0 \), are obtained
as a special case of the more general normalized CES class of LPFs by taking the limit
\( \theta \to -\infty \) (we denote this case as \( \theta = -\infty \) for simplicity). \( \pi_0 = \frac{r_0K_0}{Y_0} \) is the capital
income share at \( t_0 \). The LPF exhibits constant returns to scale.

Please note that in the normalization procedure of the CES LPFs, benchmark values
have been assigned not only to output, capital and labor \((Y_0, K_0, L_0)\), but also to the
benchmark technology \((b_0, a_0)\). In the following derivations, this benchmark technology
will be identified with the optimal technology at time \( t_0 \).

By assuming \( \theta < 0 \), or equivalently \( \sigma_{LPF} < 1 \), we concentrate on the likely case
where LPFs allow little or no substitutability between inputs. More precisely, capital
and labor are assumed to be gross complements along the LPF. Since all results derived
in this paper go through also in the limiting case of Leontief LPFs, where inputs are
perfectly complementary, the CES specification of the LPF is not strictly necessary for
the aggregate CES production function to obtain.

From the economic point of view, the assumption that there is little or no input
substitutability along the LPF is consistent with the “recipe” interpretation of particular
production techniques (where the LPF is viewed as a list of instructions on how to trans-
form inputs into output, that must be followed as closely as possible, cf. Jones, 2005),
and thus it is clearly favorable over the alternative cases \( \theta \in (0,1) \) (gross substitutability
of inputs along the LPF) and \( \theta \to 0 \) (Cobb–Douglas LPFs). For this reason, we shall concentrate on this case in the following calculations. From the analytical point of view, all these cases may be considered, though. The discussion of the other cases is included in the appendix.

The CES/Leontief specification of the LPF was used in the earlier contributions of Growiec (2008a,b), but without normalization.

**Assumption 2** *The technology menu, specified in the \((a,b)\) space, is given by equality:*

\[
H(a,b) = \left( \frac{a}{\lambda_a} \right)^\alpha + \left( \frac{b}{\lambda_b} \right)^\alpha = N, \quad \lambda_a, \lambda_b, \alpha, N > 0.
\]  

(2)

The technology menu is thus a downward-sloping curve in the \((a,b)\) space, capturing the trade-off between the available UFPs of capital and labor. At the current stage of the analysis, we just take the functional form (2) for granted, like Caselli and Coleman (2006) or Nakamura (2009).

In Section 5, however, the technology menu will be derived as a *contour line of the cumulative distribution function* of the joint bivariate distribution of capital- and labor-augmenting ideas (\(\tilde{b}\) and \(\tilde{a}\), respectively). The key point of Section 5 will be to put forward a tractable model of the R&D sector that will generate the technology menu (2) from a general class of individual (marginal) distributions of \(\tilde{b}\) and \(\tilde{a}\). Under independence of both dimensions (so that marginal distributions of \(\tilde{b}\) and \(\tilde{a}\) are simply multiplied by one another), equation (2) is obtained if and only if the marginal distributions are Weibull with the same shape parameter \(\alpha > 0\) (Growiec, 2008b):

\[
P(\tilde{a} > a) = e^{-\left( \frac{a}{\lambda_a} \right)^\alpha}, \quad P(\tilde{b} > b) = e^{-\left( \frac{b}{\lambda_b} \right)^\alpha},
\]

(3)

for \(a, b > 0\). Under such parametrization, we have \(P(\tilde{a} > a, \tilde{b} > b) = e^{-\left( \frac{a}{\lambda_a} \right)^\alpha - \left( \frac{b}{\lambda_b} \right)^\alpha}\), and thus the parameter \(N\) in eq. (2) is interpreted as \(N = -\ln P(\tilde{a} > a, \tilde{b} > b) > 0\). In what follows, we shall assume \(N\) to be constant across time, and \(\lambda_a, \lambda_b\) to grow as an outcome of factor-augmenting R&D.\(^8\)

\(^7\)Equation (2) can also be obtained for Pareto distributions of \(\tilde{a}\) and \(\tilde{b}\), provided that the pattern of dependence between both marginal distributions is modeled with the Clayton copula (Growiec, 2008a). The alternative case where \(\tilde{a}\) and \(\tilde{b}\) are independently Pareto-distributed leads to a different specification of the technology menu which will be considered separately in Section 4.

\(^8\)One could easily reparametrize the technology menu, though, fixing either \(\lambda_a\) or \(\lambda_b\) and allowing \(N\) to vary. This would also reparametrize the resultant aggregate production function: the ratio \(N/N_0\) would appear in equation (8) and the fixed parameter \((\lambda_a\) or \(\lambda_b\)) would drop out. One could also (redundantly) vary all three parameters simultaneously. See the discussion in Section 3.
From the mathematical point of view, we could also allow the shape parameter $\alpha$ in (2) to be negative. In that case, the technology menu would be reinterpreted as a contour line of a bivariate distribution derived from two independent Fréchet distributions. Since the economic interpretation favors the case with $\alpha > 0$ (see Section 5), the derivations for the case $\alpha < 0$ have been relegated to the appendix.

An important caveat is that if $\tilde{a}$ and $\tilde{b}$ are Weibull-distributed but dependent, or independent but following some other distribution than Pareto or Weibull, the resultant aggregate production does not belong to the CES class. It is also vital that both marginal Weibull distributions share the same shape parameter $\alpha$: if labor- and capital-augmenting ideas are independently Weibull distributed, but with different shape parameters, then the resultant aggregate production function does not belong to the CES class either. Fortunately, as Section 5 shows, under arguably general conditions the model of directed R&D discussed there will yield the same value of $\alpha = 1$.

The analytical form of the technology menu postulated in equation (2) has been used previously by Caselli and Coleman (2006), Growiec (2008a,b), and Nakamura (2009), but with the unnecessary restriction that $\lambda_a$ and $\lambda_b$ are always either constant or increasing proportionately.

**Assumption 3** Firms choose the technology pair $(a, b)$ optimally, subject to the current technology menu, such that their profit is maximized:

$$
\max_{a,b} \left\{ Y_0 \left( \pi_0 \left( \frac{bK}{b_0K_0} \right)^\theta + (1 - \pi_0) \left( \frac{aL}{a_0L_0} \right)^\theta \right) \right\} \quad \text{s.t.} \quad \left( \frac{a}{\lambda_a} \right)^\alpha + \left( \frac{b}{\lambda_b} \right)^\alpha = N.
$$

(4)

We note that factor remuneration $rK + wL$, taken account in the firms’ profit maximization problem, does not depend on the chosen technology pair $(a, b)$ so it can be safely omitted from the above optimization problem.\(^9\) The same assumption was made by Jones (2005), Growiec (2008a,b), and Nakamura (2009).

Finally, second order conditions require us to assume that $\alpha > \theta$, so that the interior stationary point of the above optimization problem is a maximum. The proof of this is included in the appendix. Furthermore, we also need to assume that $\alpha - \theta - \alpha\theta > 0$ so that the resultant aggregate production function is concave with respect to $K$ and $L$.

\(^9\)In the case of Leontief LPFs, optimization requires $\frac{bK}{b_0K_0} = \frac{aL}{a_0L_0}$.
Both these conditions are satisfied automatically in the case $\alpha > 0 > \theta$, on which we concentrate here. As we shall see shortly, in such case, capital and labor will always be gross complements along the aggregate production function.

### 2.2 Technology choice and the aggregation result

Solving the maximization problem set up above yields direct results on the firm’s optimal technology choices. First, at time $t_0$, when $K = K_0$, $L = L_0$, $Y = Y_0$, $\lambda_a = \lambda_{a0}$, $\lambda_b = \lambda_{b0}$, it is easily verified that the optimal choice is:

$$a_0^* = (N(1 - \pi_0))^{\frac{1}{\alpha}} \lambda_{a0}, \quad b_0^* = (N\pi_0)^{\frac{1}{\alpha}} \lambda_{b0},$$

(5)

where $\lambda_{a0}$ and $\lambda_{b0}$ are the values of $\lambda_a$ and $\lambda_b$ at time $t_0$, respectively. The above values of $a_0^*$ and $b_0^*$ will be used as $a_0$ and $b_0$ in the normalization at the local level in all subsequent derivations.

For any other moment in time $t \neq t_0$, the optimal technology choices are:

$$\left(\frac{a}{a_0}\right)^* = \frac{\lambda_a}{\lambda_{a0}} \left(\pi_0 \left(\frac{\lambda_b \lambda_{a0} KL_0}{\lambda_a \lambda_{b0} LK_0}\right)^{\frac{\alpha^b}{\alpha}} + 1 - \pi_0\right)^{-\frac{1}{\alpha}},$$

(6)

$$\left(\frac{b}{b_0}\right)^* = \frac{\lambda_b}{\lambda_{b0}} \left(\pi_0 + (1 - \pi_0) \left(\frac{\lambda_b \lambda_{a0} KL_0}{\lambda_a \lambda_{b0} LK_0}\right)^{-\frac{\alpha^b}{\alpha^b - \alpha}}\right)^{-\frac{1}{\alpha}},$$

(7)

where $\frac{\alpha^b}{\alpha - \theta}$ is substituted with $-\alpha$ in the case of Leontief LPFs ($\theta = -\infty$).

Inserting these optimal technology choices into the LPF, we obtain the following result.

**Proposition 1** If Assumptions 1-3 hold, then the aggregate production function takes the normalized CES form:

$$Y = Y_0 \left(\pi_0 \left(\frac{\lambda_b}{\lambda_{b0}} \frac{K}{K_0}\right)^{\frac{\alpha}{\alpha - \theta}} + (1 - \pi_0) \left(\frac{\lambda_a}{\lambda_{a0}} \frac{L}{L_0}\right)^{\frac{\alpha^b}{\alpha^b - \alpha}}\right)^{\frac{\alpha - \theta}{\alpha^b - \alpha}}.$$ 

(8)

Hence, the normalized CES result obtains both in the case of CES and Leontief LPFs.

**Proof** (and generalization to $n$ inputs): see the appendix.

It is worthwhile to comment on each of the parameters of the aggregate normalized CES production function, because they all have sound interpretations:
• the substitutability parameter is $\rho = \frac{a\theta}{a-\theta}$ (or $\rho = -\alpha$ in the case of Leontief LPFs). The aggregate elasticity of substitution is thus $\sigma = \frac{1}{1-\rho} = \frac{a-\theta}{a-\theta-a\theta} > 0$ (or $\sigma = \frac{1}{1+\alpha}$ in the case of Leontief LPFs). It is verified that $\theta < \rho < 0$ and thus $\sigma_{LPF} = \frac{1}{1-\theta} < \sigma < 1$. Hence, endogeneous technology choice unambiguously increases the substitutability between production factors as compared to the LPF, but the elasticity of substitution nevertheless remains bounded from above by unity, characteristic for the Cobb–Douglas specification.\(^{10}\)

• the distribution parameter is $\pi_0 = \frac{r_0 K_0}{Y_0}$, whereas the multiplicative constant term is $Y_0$. Hence, thanks to normalization, both these parameters are equal to the respective parameters of the LPF;

• the constant parameter $N$ does not appear in the aggregate production function,

• the capital-augmenting factor $b$ present in the LPF is replaced by the capital-augmenting parameter of the technology menu $\lambda_b$ in the aggregate production function, both at time $t_0$ and at any time $t \neq t_0$. The same applies to the labor-augmenting factor $a$ and the respective parameter $\lambda_a$.\(^{11}\)

Hence, all growth in $\lambda_a$ and $\lambda_b$ (with $N$ kept intact), obtained thanks to directed R&D, will ultimately appear as a multiplicative term in front of the respective factor of production in the aggregate production function. As it will be shown in the following section, however, growth in $\lambda_a$ or $\lambda_b$ ought not to be confused with the actual factor-augmenting technical change, i.e., growth in $a$ and $b$: already from equations (6)–(7), one sees that these two types of entities are generally not proportional to one another, unless additional conditions are met.

We also note the following straightforward corollary.

**Corollary 1** Assuming that factors are priced at their marginal product, the capital and

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\(^{10}\)Capital and labor could be gross substitutes in the aggregate production function only if $\theta > 0$ or $\alpha < 0$ (and a few other auxiliary conditions are met). For a discussion of these cases, please refer to the appendix.

\(^{11}\)The last two findings are a corollary from the fact that the technology menu defined in Assumption 2 is a curve in a two-dimensional space, parametrized by three parameters $\lambda_a, \lambda_b, N$, and $N$ is kept constant by assumption.
labor income shares are equal to, respectively:

\[
\pi = \frac{rK}{Y} = \frac{\pi_0 \left( \frac{\lambda_b K}{\lambda_{0b} K_0} \right)^{\alpha \theta}}{\pi_0 \left( \frac{\lambda_b K}{\lambda_{0b} K_0} \right)^{\alpha \theta} + (1 - \pi_0) \left( \frac{\lambda_a L}{\lambda_{0a} L_0} \right)^{\alpha \theta}}, 
\]  

(9)

\[
1 - \pi = \frac{wL}{Y} = \frac{(1 - \pi_0) \left( \frac{\lambda_a L}{\lambda_{0a} L_0} \right)^{\alpha \theta}}{\pi_0 \left( \frac{\lambda_b K}{\lambda_{0b} K_0} \right)^{\alpha \theta} + (1 - \pi_0) \left( \frac{\lambda_a L}{\lambda_{0a} L_0} \right)^{\alpha \theta}}. 
\]  

(10)

The above factor share formulas are instructive as regards the expected direction of their change over time: this direction is strictly determined by the growth rate of \( \lambda_b K \) relative to \( \lambda_a L \). If both growth rates are equal, the capital income share will remain constant at \( \pi_0 \). If \( \lambda_b K \) grows faster, then due to gross complementarity of inputs along the aggregate production function \( (\alpha \theta < 0) \), capital’s income share will gradually fall to zero over time; conversely, it will gradually rise towards unity if \( \lambda_a L \) grows faster. Hence, endogeneous technology choice does not overturn any of the standard results identified in earlier literature.\(^{12}\)

We may also note that if technology adoption required time, so that there were a nonzero time lag between any exogeneous shift in the firm’s factor endowments and the adoption of the new optimal technology by this firm (cf. León-Ledesma and Satchi, 2011), then the elasticity of substitution along the LPF, \( \sigma_{LPF} \), could be reinterpreted as the short-run elasticity of substitution, whereas \( \sigma \) would then be understood as the long-run elasticity of substitution. The implication that \( \sigma_{LPF} < \sigma \) is consistent with economic interpretations provided by, e.g., Jones (2005) and León-Ledesma and Satchi (2011). The conclusion that \( \sigma_{LPF} < 1 \) as well as \( \sigma < 1 \) is, in turn, strongly supported by empirical evidence, reviewed by Chirinko (2008) and León-Ledesma et al. (2010).\(^{13}\)

\(^{12}\)An obvious remark here is that the above factor share formulas are invalidated once one allows for imperfect competition. However, if it is introduced via the Dixit–Stiglitz model of monopolistic competition, implying constant markups over marginal costs, then this change would merely re-scale factor income shares, without altering any of the results on their dynamics.

\(^{13}\)From the purely analytical point of view, our framework allows for the possibility of gross substitutability of inputs along the aggregate production function as well. Please consult the appendix.
3 Implications for the direction of technical change

Given the static character of the endogeneous technology choice framework discussed in the previous section, it is straightforward to embed it in dynamic growth models. To keep things simple, one could, for instance, assume additionally that there are no interactions between technology choice and factor demand on the side of firms. In such case, the aggregate production function derived in equation (8) would enter the dynamic model directly. Such a growth model could be closed, e.g., by allowing directed R&D to increase $\lambda_a$ and $\lambda_b$ endogeneously (cf. Acemoglu, 2003). Then, depending on the particular assumptions of the embedding growth model, generally any direction of R&D could be obtained in equilibrium.\footnote{Albeit perhaps only a few of these models would be useful for an economist.}

In the following subsections, we shall discuss a few examples of growth models that can be used as such embedding structures. Our attention will be focused largely on the distinction between the direction of R&D, captured by the relative growth rates of $\lambda_a$ and $\lambda_b$, and the actual direction of technical change, measured by the relative growth rates of $a$ and $b$.

3.1 Balanced growth path, Harrod–neutral R&D, and purely-augmenting technical change

It is frequently postulated in the economic growth literature – partly due to the plentiful established “stylized facts” and partly due to analytical convenience – that the aggregate economy should follow a balanced growth path (BGP), or at least converge to it in the long run. It should be emphasized, however, that the existence of a BGP is a very restrictive, \textit{knife-edge} assumption (cf. Growiec, 2007): in the case of the neoclassical growth model, it requires that either the aggregate production function is Cobb–Douglas or technical change is purely labor-augmenting (Uzawa, 1961).

A seminal example of a growth model with (endogeneous) directed R&D, that is able to generate a BGP despite being based upon a CES production function, is due to Acemoglu (2003). This model can be straightforwardly used as an embedding structure over our endogeneous technology choice framework. It requires the equations of motion of factor-augmenting shifts in the technology menu to be captured by linear equations...
of form:
\[ \dot{\lambda}_a = f_a(\ell_a)\lambda_a, \]  
\[ \dot{\lambda}_b = f_b(\ell_b)\lambda_b, \]  
where \( \ell_a \) and \( \ell_b \) are the fractions of population (total employment, hours worked) engaged in labor- and capital-augmenting R&D, respectively, and \( f_a, f_b \) are some smooth increasing functions. Such a model is scale-free. We also assume the usual capital’s equation of motion:
\[ \dot{K} = sY - \delta K, \quad \delta > 0, \]
where \( s \) is the (potentially endogeneous) savings rate of the aggregate economy.

Analyzing the model’s implications, both in the social planner allocation and the decentralized equilibrium à la Acemoglu, reveals that in the long run, the economy will converge to a balanced growth path where:
\[ \hat{\dot{y}} = \hat{\dot{k}} = \hat{\dot{a}} = \hat{\dot{\lambda}_a} = f(\ell_a^*) > 0, \]  
\[ \hat{\dot{b}} = \hat{\dot{\lambda}_b} = 0, \]
and thus:

(i) R&D is Harrod–neutral, i.e., directed toward labor-augmenting developments only \( (\dot{\lambda}_b = 0) \),

(ii) technical change is also Harrod–neutral, i.e., purely labor-augmenting \( (\dot{b} = 0) \),

(iii) factor income shares are constant at \( \pi_0 \) and \( 1 - \pi_0 \), respectively, despite the fact that the production function is not Cobb–Douglas.

Hence, in this very specific framework with linear technology equations in both R&D sectors and no intrasectoral spillovers between both sectors, technical change must follow the direction of R&D on one-to-one basis.

Further analysis reveals that the equations of motion of capital- and labor-augmenting developments \( \lambda_a \) and \( \lambda_b \) (11)–(12), could be made slightly more general without altering any of the aforementioned predictions. This would happen if one allowed for mutual spillovers between both R&D sectors, yet also imposed a particular knife-edge condition.

---

\[^{15}\text{We use the notation: } y \equiv Y/L, k \equiv K/L, \text{ and } \dot{x} \equiv \dot{x}/x \text{ for any variable } x > 0.\]
on their strength (measured by partial elasticities). This result has been obtained by Li (2000), for a somewhat different two-R&D-sector model, which is however identical to the current one in its reduced form (that is, after stripping its solution to the form of a system of dynamic equations governing the dynamics of its state variables).

What is much more important here, however, is that for every other reduced-form specification of the encompassing growth model, the above BGP result will fail. Hence, in the typical (non-knife-edge) case, technical change will not be purely labor-augmenting, it will not reflect the direction of R&D, and factor income shares will not be constant across time (Growiec, 2007). Let us now illustrate that phenomenon with an interesting example.

### 3.2 Hicks–neutral R&D

The second benchmark case of an encompassing growth model to be discussed here is the model of Hicks–neutral R&D, where R&D always expands the technology menu proportionally, without any bias towards any of the production factors. On the one hand, such a model is clearly just as parsimonious as the Harrod–neutral R&D model discussed above: arriving at this particular case also requires one to make a certain knife-edge assumption. On the other hand, however, it provides us with a second reasonable benchmark to which we can compare the general results for all non-neutral cases.

Hicks–neutral R&D has been considered in very similar endogeneous technology choice frameworks by Caselli and Coleman (2006) and Growiec (2008a). These authors have made the implicit assumption that factor-augmenting idea distributions \( \tilde{a} \) and \( \tilde{b} \) evolve proportionally, however, so that the ratio \( \lambda_a / \lambda_b \) is constant, and thus \( \dot{\lambda}_a = \dot{\lambda}_b \) for all times \( t \).\(^{16}\) An example of a model where R&D is Hicks–neutral for all \( t \) can be written as follows:

\[
\begin{align*}
\dot{\lambda}_a &= f(\ell_a, \ell_b)\lambda_a^{\alpha+1}\lambda_b^{\beta}, \\
\dot{\lambda}_b &= f(\ell_a, \ell_b)\lambda_a^{\alpha}\lambda_b^{\beta+1},
\end{align*}
\]

\(^{16}\)Growiec (2008a) assumed that \( \lambda_a \) and \( \lambda_b \) were fixed, and proxied technological progress by growth in \( N \) instead. This is equivalent, however, in terms of the evolution of the technology menu over time, to keeping \( N \) fixed and varying \( \lambda_a \) and \( \lambda_b \) proportionately. An analogous assumption was made by Caselli and Coleman (2006) in the cross-sectional context: they allowed only \( N \) to vary across countries, but \( \lambda_a \) and \( \lambda_b \) were kept fixed.
and hence by assumption $\hat{\lambda}_a = \hat{\lambda}_b$.\textsuperscript{17} We shall also use the usual capital’s law of motion (13) again.

As argued above, Hicks–neutral R&D precludes the existence of a balanced growth path. More surprisingly, however, it also implies that technical change in the aggregate economy, determined jointly by the direction of R&D and firms’ endogeneous technology choices, is not Hicks–neutral. In particular, under the assumptions that (i) Hicks–neutral R&D improves both UFPs at the same constant rate, so that $\hat{\lambda}_a = \hat{\lambda}_b \equiv g > 0$, (ii) the economy is able to maintain positive growth rates of physical capital per worker $k$ until infinite time, with $\lim_{t \to \infty} k(t) = +\infty$, technical change will augment both factors of production in the long run, according to:

\begin{align*}
\lim_{t \to \infty} \hat{y}(t) &= g, \\
\lim_{t \to \infty} \hat{a}(t) &= g, \\
\lim_{t \to \infty} \hat{b}(t) &= g + \frac{\theta}{\alpha - \theta} \lim_{t \to \infty} \hat{k}(t) \quad \Rightarrow \quad \lim_{t \to \infty} \hat{b}(t) \in \left[ \left( \frac{\alpha}{\alpha - \theta} \right) g, g \right].
\end{align*}

These results have been obtained by taking limits of optimal technology choices (6)–(7) under the assumption that $k(t) \to +\infty$. We have also used the inequality $\lim_{t \to \infty} \hat{k}(t) \leq \lim_{t \to \infty} \hat{y}(t)$, because in the opposite case, $y(t)/k(t)$ would be falling towards zero, ultimately violating the capital’s equation of motion.\textsuperscript{18}

Hence, in the case of Hicks-neutral R&D, technological change augments both factors of production in the long run. Capital-augmenting technical change remains positive forever, too, in contrast to the findings based on the Cobb–Douglas production function (Jones, 2005).

On the other hand, even if R&D is Hicks–neutral, endogeneous technology choice still introduces a bias in the direction of technical change in favor of labor, the scarce non-accumulable input. Firms decide optimally to increase the UFP of labor faster than that of capital in order to adjust to the ongoing changes in factor proportions, which are in favor of capital. This is natural given gross complementarity of both inputs.

Furthremore, under Hicks–neutral R&D, the capital income share $\pi$ is bound to fall gradually towards zero, provided that the rate of capital accumulation remains positive.

\textsuperscript{17}\hspace{1em}In a somewhat larger (yet, still very specific) class of models, R&D will be Hicks–neutral in the limit of $t \to \infty$. The long-run results obtained within this section hold for such models as well.

\textsuperscript{18}\hspace{1em}Assuming furthermore that $\lim_{t \to \infty} \hat{k}(t) = \lim_{t \to \infty} \hat{y}(t) = g$, it follows that $\lim_{t \to \infty} \hat{b}(t) = \left( \frac{\alpha}{\alpha - \theta} \right) g > 0$.  

16
in the long run.

3.3 All other directions of R&D

For all other cases of directed R&D, implying \( \lambda_a \neq \lambda_b \) and \( \lambda_b \neq 0 \) over the long run, we obtain the following asymptotical results.

- If \( \lambda_b K \) grows asymptotically faster than \( \lambda_a L \), then \( \dot{y}(t) \to \lambda_a(t), \dot{a}(t) \to \lambda_a(t) \) and \( \dot{b}(t) \to \lambda_b(t) + \frac{\theta}{\alpha - \theta} \dot{k}(t) \) with \( t \). The capital income share falls towards zero over time.

- If \( \lambda_b K \) grows asymptotically slower than \( \lambda_a L \), then \( \dot{y}(t) \to \lambda_b(t) + \dot{k}(t), \dot{a}(t) \to \lambda_a(t) - \frac{\theta}{\alpha - \theta} \dot{k}(t) \) and \( \dot{b}(t) \to \lambda_b(t) \) with \( t \). The capital income share increases towards unity over time.

- If \( \lambda_b K \) grows asymptotically at the same pace as \( \lambda_a L \), then \( \dot{y}(t) \to \lambda_a(t) + \dot{k}(t), \dot{a}(t) \to \lambda_a(t) \) and \( \dot{b}(t) \to \lambda_b(t) \) with \( t \). The capital income share tends to a constant.

Hence, we see that in the general case, the direction of R&D and the direction of technical change are different from one another. Moreover, the temporal evolution of optimal technology choices and factor income shares is determined by comparing the growth rates of \( \lambda_b K \) and \( \lambda_a L \), i.e., of capital and labor expressed in efficient units evaluated along the aggregate production function.

4 Cobb–Douglas aggregate production function

Let us now return to our main “endogeneous technology choice” framework, presented in Section 2, and demonstrate how it could be used to derive the aggregate Cobb–Douglas production function (cf. Jones, 2005). The key change in assumptions that is required to produce this result relates to the distribution of capital- and labor-augmenting ideas

19 Given that \( \dot{k}(t) \leq \dot{y}(t) \) for sufficiently large \( t \), this case can only be obtained if \( \lambda_b(t) > 0 \).

20 If \( \dot{k}(t) = \dot{y}(t) \) in the long run and \( \lambda_b > 0 \), then such a model would imply explosive dynamics, potentially achieving infinite output in finite time.

21 If \( \dot{k}(t) = \dot{y}(t) \) in the long run, then this case can appear only if \( \lambda_b(t) \to 0 \), which boils down to the balanced growth path case discussed above.
and thus the shape of the technology menu; everything else is preserved. Even though in Section 5, we are going to argue that this change in assumptions is actually quite misleading, and the aggregate CES production function (with gross complementarity of inputs) is in fact a more plausible alternative, the Cobb–Douglas case is nevertheless a useful benchmark for comparisons because it is so frequently used in the literature.

4.1 Modification of the framework

Let us now replace Assumption 2 with the following one:

**Assumption 4 (modification of Assumption 2)** The technology menu, defined in the \((a,b)\) space, is given by the equality:

\[
H(a,b) = \left(\frac{a}{\lambda_a}\right)^{\phi_L} \left(\frac{b}{\lambda_b}\right)^{\phi_K} = N, \quad \phi_K, \phi_L > 0. \tag{21}
\]

The shape of the technology menu given by equation (21) is consistent with the assumption that \(\tilde{a}\) and \(\tilde{b}\) are independently Pareto-distributed, with shape parameters \(\phi_L\) and \(\phi_K\), respectively:

\[
P(\tilde{a} > a) = \left(\frac{\lambda_a}{a}\right)^{\phi_L}, \quad P(\tilde{b} > b) = \left(\frac{\lambda_b}{b}\right)^{\phi_K}, \tag{22}
\]

for \(a > \lambda_a\) and \(b > \lambda_b\). In such case, \(N = \frac{1}{P(\tilde{a} > a, \tilde{b} > b)} > 1\). Just like in Section 2, we assume \(N\) to be fixed, and allow \(\lambda_a\) and \(\lambda_b\) to rise over time thanks to directed R&D.

The same functional form of the technology menu was assumed by Jones (2005), but with the unnecessary restriction of proportional (Hicks–neutral) augmentation of the technology menu, which is now relaxed.

4.2 Technology choice and the aggregation result

It is easily verified that at time \(t_0\), when \(K = K_0, L = L_0, Y = Y_0, \lambda_a = \lambda_{a0}, \lambda_b = \lambda_{b0}, \) the optimal technology choice is indeterminate, provided that

\[
\pi_0 = \frac{r_0 K_0}{Y_0} = \frac{\phi_K}{\phi_L + \phi_K}. \tag{23}
\]

This restriction means that \(\pi_0\), the capital income share at \(t_0\) (and in fact at all other times as well), must be equal to \(\frac{\phi_K}{\phi_L + \phi_K}\). Thus, \(\pi_0\) ceases to be a free parameter in the
current case, and \(a_0\) becomes a free parameter instead \((b_0\) is then calculated according to \(\lambda a_0 \phi_L \phi_L + \phi_K \phi_K = N\)).

Furthermore, at any other moment in time \(t \neq t_0\), and given \(a_0\) and \(b_0\), the optimal technology choices are:

\[
\left(\frac{a}{a_0}\right)^* = \left(\frac{\lambda a}{\lambda a_0}\right)^{\phi_L + \phi_K} \left(\frac{\lambda b}{\lambda b_0}\right)^{\phi_K + \phi_K} \left(\frac{KL_0}{LK_0}\right)^{-\phi_L - \phi_K} \quad (24)
\]

\[
\left(\frac{b}{b_0}\right)^* = \left(\frac{\lambda a}{\lambda a_0}\right)^{\phi_L + \phi_K} \left(\frac{\lambda b}{\lambda b_0}\right)^{\phi_K + \phi_K} \left(\frac{KL_0}{LK_0}\right)^{-\phi_L - \phi_K} \quad (25)
\]

Inserting these optimal technology choices into the LPF, we obtain the following aggregation result.

**Proposition 2** If Assumptions 1, 3, and 4 hold, then the aggregate production function takes the normalized Cobb–Douglas form:

\[
Y = Y_0 \left(\frac{\lambda a}{\lambda a_0}\right)^{\phi_L} \left(\frac{\lambda b}{\lambda b_0}\right)^{\phi_K} \left(\frac{KL_0}{LK_0}\right)^{\phi_K} \left(\frac{L}{L_0}\right)^{\phi_L} \cdot (26)
\]

**Proof** (and generalization to \(n\) inputs): see the appendix. ■

The interpretation of the parameters of the aggregate Cobb–Douglas production function is the following:

- the distribution parameter of the aggregate Cobb–Douglas production function, equal to the capital’s partial elasticity and the (constant) capital income share, takes the value \(\pi_0 = \frac{\phi_K}{Y_0} = \frac{\phi_K}{\phi_L + \phi_K}\),

- partial elasticities of capital and labor in the aggregate production function are proportional to the shape parameters of the Pareto distributions of their respective factor-augmenting technologies and sum up to one (guaranteeing constant returns to scale),

- the multiplicative constant term is \(Y_0\). Thanks to normalization, it is thus exactly equal to the multiplicative constant term of the LPF,

- the constant parameter \(N\) does not appear in the aggregate production function,\(^{22}\)

\(^{22}\)Again, one could easily reparametrize the technology menu, fixing either \(\lambda a\) or \(\lambda b\) and allowing \(N\) to vary across time. In such case, the ratio \(N/N_0\) (which now drops out) would appear in equation (26).
the capital-and labor-augmenting parameters of the technology menu, \( \lambda_b \) and \( \lambda_a \) respectively, enter the aggregate production function multiplicatively, taken to their respective powers \( \phi_K \) and \( \phi_L \). Growth in aggregate output is thus invariant to the direction of R&D.

4.3 Direction of technical change

It is also easily verified that under endogeneous technology choice, the Cobb–Douglas case provides very specific implications for the direction of technical change. To see them, log-differentiate equations (24)–(26) with respect to time and compare terms to obtain:

\[
\dot{a} = \dot{y} = \frac{\phi_L}{\phi_L + \phi_K} \dot{\lambda}_a + \frac{\phi_K}{\phi_L + \phi_K} \dot{\lambda}_b + \frac{\phi_K}{\phi_L + \phi_K} \dot{k},
\]

(27)

\[
\dot{b} = \dot{y} - \dot{k} = \frac{\phi_L}{\phi_L + \phi_K} \dot{\lambda}_a + \frac{\phi_K}{\phi_L + \phi_K} \dot{\lambda}_b - \frac{\phi_L}{\phi_L + \phi_K} \dot{k}.
\]

(28)

Hence, it follows that in the Cobb–Douglas case, no matter what the direction of R&D is, i.e., irrespective of the values of \( \dot{\lambda}_a \) and \( \dot{\lambda}_b \), firms will always adjust the labor-augmenting technology on one-to-one basis to changes in output per worker \( y \), and capital-augmenting technology will be, accordingly, always adjusted one-to-one to changes in output per unit of capital \( y/k \). Hence, as shown by Jones (2005), technological change must be purely labor-augmenting along the balanced growth path, where the output–capital ratio \( y/k \) is constant.

Assuming that factors of production are remunerated according to their marginal product, the capital income share is now fixed at \( \pi_0 = \frac{\phi_K}{\phi_L + \phi_K} \), and the labor income share is fixed at \( 1 - \pi_0 = \frac{\phi_L}{\phi_L + \phi_K} \), for all times \( t \).

5 The Weibull distribution in R&D productivity

Having discussed the key properties of the current “endogeneous technology choice” model where the aggregate production function is derived as a convex hull of local production techniques, with UFPs selected from the given technology menu, let us now justify the functional form of this menu, taken for granted in Assumption 2. This will be done using a novel, analytically tractable model of two independent R&D sectors,
producing capital- and labor-augmenting innovations, respectively. It bears some similarity with the framework discussed in Appendix D of Growiec (2008a), but has a few unique distinguishing features. The model will be characterized in the two following subsections.

5.1 Distributions of complex ideas

The point of departure of the current model is the assumption that ideas are inherently complex and consist of a large number of complementary components. Formally, this can be written down in the following way.

**Assumption 5** The (capital- or labor-augmenting) R&D sector consists of an infinity of researchers located along the unit interval $I = [0, 1]$. At each instant $t$, every researcher $i \in I$ determines the quality of her innovation ($\tilde{b}_i$ or $\tilde{a}_i$, respectively) by taking the minimum over $n \in \mathbb{N}$ independent draws from the elementary idea distribution with cdf $F$. The distribution $F$ has positive density on $[w, v)$, where $v$ can be infinite, and zero density otherwise, and satisfies the condition

$$\lim_{p \to 0^+} \frac{F(w + px)}{F(w + p)} = x^\alpha$$

for all $x > 0$ and a certain $\alpha > 0$.

The parameter $n$ in the above assumption captures the number of constituent components of any given (composite) idea, and thus measures the complexity of any state-of-the-art technology. Allowing for such complexity puts the current framework in stark contrast to earlier studies (such as Jones, 2005, or Growiec, 2008a) where the quality of ideas was determined via a single draw from the elementary idea distribution $F$.\footnote{Jones (2005) viewed the technology menu as a convex hull of a finite number of ideas, say $M$. Hence, in the limit $M \to \infty$, this menu took the form of a contour line of a Fréchet distribution, which is the limiting distribution of the maximum of $M$ independent draws from a distribution $F$ that is bounded from below. This assumption has been later replaced, both in Growiec (2008a) and in the current paper, with Assumption 6. At this point, one should note that Jones (2005) was preoccupied with the distribution of the maximum across ideas and here we are considering the minimum across components of each idea. Across ideas, it is still the best ones that matter.}

Moreover, the assumption that the quality of an innovation is the minimum (a Leontief function) of a range of $n$ independent draws from the distribution $F$ reflects the
view that the components of an idea are complementary to one another (Kremer, 1993; Blanchard and Kremer, 1997; Jones, 2011). More precisely, we consider the extreme case here, where they are perfectly complementary, and thus the actual productivity of a complex idea is determined by the productivity of its “weakest link” (or “bottleneck”). Clearly, this need not hold exactly in reality, since certain deficiencies of design can often be covered by advantages in different respects. However, the example of the explosion of the space shuttle *Challenger* due to a failure of an inexpensive O-ring, put forward by Kremer (1993), is perhaps the best possible illustration of the potentially complementary character of components of complex ideas.

Letting the technology complexity $n$ be arbitrarily large, we obtain the following result:

**Proposition 3** If Assumption 5 holds, then as $n \to \infty$, the minimum of $n$ independent random draws from the distribution with cdf $\mathcal{F}$, after appropriate normalization, converges in distribution to the Weibull distribution with the shape parameter $\alpha$:

$$[1 - \mathcal{F}(xp_n + w)]^n \overset{d}{\longrightarrow} e^{-\left(\frac{1}{\lambda}\right)^\alpha},$$

where $w = \inf\{x \in \mathbb{R} : \mathcal{F}(x) > 0\}$, $p_n = \frac{1}{\lambda} \left(\mathcal{F}^{-1}\left(\frac{1}{n}\right) - w\right)$ and the free parameter $\lambda > 0$ is assumed to be proportional to the mean of the underlying distribution $\mathcal{F}$.

**Proof.** The proposition follows directly from the Fisher–Tippett–Gnedenko extreme value theorem, applied to the distribution $\mathcal{F}$ (Theorem 1.1.3 in de Haan and Ferreira, 2006, rephrased so that it captures the minimum instead of maximum). From the theorem specifying the domain of attraction of the Weibull distribution (Theorem 1.2.1 in de Haan and Ferreira, 2006; Section 1.3 in Kotz and Nadarajah, 2000), we obtain the necessary and sufficient conditions for the complementarity mechanism to work. ■

From the mathematical point of view, the parameter $\lambda$ is superfluous and can be normalized to unity without loss of generality, by a simple re-normalization of the sequence $p_n$ as $\tilde{p}_n = p_n \cdot \lambda = \mathcal{F}^{-1}\left(\frac{1}{n}\right) - w$. In the case of the currently discussed R&D model, the distinction between $p_n$ and $\lambda$ is important, however, because it allows R&D activity to influence the means of the distributions of $\tilde{a}$ and $\tilde{b}$ at any moment in time: it is precisely $\lambda$ which pins down the mean of the limiting Weibull distribution.

Hence, turning to our original distinction between capital- and labor-augmenting R&D, let us now distinguish between $\lambda_a$ determining the mean of $\tilde{a}$, and $\lambda_b$ pinning
down the mean of $\tilde{b}$. In the limit of $n \to \infty$, we obtain the following generic results:

$$E\tilde{a} = \lambda_a \Gamma \left(1 + \frac{1}{\alpha}\right), \quad E\tilde{b} = \lambda_b \Gamma \left(1 + \frac{1}{\alpha}\right),$$

(31)

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the Euler’s Gamma function.

A few meaningful applications of Proposition 3 have been summarized in Table 1. This table indicates that the Weibull result can be obtained for a wide range of distributions, including the ones frequently discussed in the related literature. It also provides a clear explanation of the relationship between $\lambda$ and the characteristics of the underlying productivity distribution $F$, and announces the resulting values of the parameter $\alpha$.

The main message drawn from the results presented in Table 1 is that we can model directed R&D, affecting the mean of the underlying distribution $F$ and thus $\lambda_a$ and $\lambda_b$, in an arbitrary way; if only the conditions specified in Assumption 5 hold, then the Weibull result in the limit of $n \to \infty$ will always go through. If, on top of that, the shape parameters $\alpha$ of capital- and labor-augmenting developments happen to be equal to one another, then the aggregate normalized CES production function result will always follow, too.

Fortunately for the last statement, the implied parameter $\alpha$ is found to be unitary for a wide range of distributions $F$, and therefore the condition that $\alpha$ is equal for both capital- and labor-augmenting ideas is indeed quite plausible. Furthermore, if $\alpha = 1$, then also $E\tilde{a} = \lambda_a$ and $E\tilde{b} = \lambda_b$, which makes the link between the underlying distribution $F$ and the limiting Weibull distribution (which in such case specializes to the exponential distribution) especially apparent. We note the following corollary.

**Corollary 2** If the underlying idea distributions $F$ are Pareto, uniform or truncated Gaussian, then $\alpha = 1$ and thus the limiting idea distribution is exponential. In such case, the elasticity of substitution of the aggregate CES production function is equal to $\sigma = \frac{1-\theta}{1-2\theta} \in \left[\frac{1}{2}, 1\right)$, increasing from $\frac{1}{2}$ in the case of Leontief LPFs to unity in the limiting case of Cobb–Douglas LPFs.
Table 1: Selected distributions $\mathcal{F}$ such that for $X_1, \ldots, X_n \sim \mathcal{F}$, $\min\{X_1, \ldots, X_n\}$ converges in distribution to the Weibull distribution as $n \to \infty$.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Cdf for $x \in [w, v)$</th>
<th>Lower bound</th>
<th>Postulated $p_n$</th>
<th>Implied $\lambda$</th>
<th>Implied $\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pareto($\phi$)</td>
<td>$\mathcal{F}(x) = 1 - \left(\frac{v}{x}\right)^\phi$</td>
<td>$w = \gamma_x$</td>
<td>$p_n = \left(1 - \frac{1}{n}\right)^{-\frac{1}{\phi}}$</td>
<td>$\lambda = \gamma_x$</td>
<td>$\alpha = 1$</td>
</tr>
<tr>
<td>Uniform $U([w, v])$</td>
<td>$\mathcal{F}(x) = \frac{x-w}{v-w}$</td>
<td>given $w$</td>
<td>$p_n = \frac{1}{n}$</td>
<td>$\lambda = v - w$</td>
<td>$\alpha = 1$</td>
</tr>
<tr>
<td>Truncated $N(\mu, \sigma)$</td>
<td>$\mathcal{F}(x) = \frac{\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi\left(\frac{w-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{w-\mu}{\sigma}\right)}$</td>
<td>given $w$</td>
<td>$p_n = \bar{p}_n$</td>
<td>$\lambda = \mu$</td>
<td>$\alpha = 1$</td>
</tr>
<tr>
<td>Weibull($\alpha, \lambda$)</td>
<td>$\mathcal{F}(x) = 1 - e^{-\left(\frac{x}{\lambda}\right)^\alpha}$</td>
<td>$w = 0$</td>
<td>$p_n = -\frac{1}{\alpha} \ln(1 - \frac{1}{n})$</td>
<td>given $\lambda$</td>
<td>given $\alpha$</td>
</tr>
</tbody>
</table>

Notes: (i) to obtain convergence to the Weibull distribution, one may equivalently take $p_n = \frac{1}{n}$ in the Pareto case, and $p_n = n^{-\frac{1}{\alpha}}$ in the Weibull case; (ii) we used the notation

$$\bar{p}_n = 1 - \frac{w}{\bar{\mu}} + \frac{\sigma}{\mu} \Phi^{-1}\left(\left[1 - \Phi\left(\frac{w-\mu}{\sigma}\right)\right] \frac{1}{n} + \Phi\left(\frac{w-\mu}{\sigma}\right)\right).$$

The mean of a random variable drawn from the truncated Gaussian distribution increases both with the mean of the original distribution $\mu$ and the truncation point $w$, according to the formula $EX = \mu + \frac{\sigma \Phi\left(\frac{w-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{w-\mu}{\sigma}\right)}$. We have chosen technological change in $\lambda$ to affect $\mu$, but we might have alternatively chosen it to affect $w$, or some mixture of both.
Please also note that a number of well-known classes of distributions have not been included in Table 1, because they do not satisfy Assumption 5. First of all, the support of the distribution must be bounded from below, which rules out distributions defined on the whole \( \mathbb{R} \) such as the Gaussian. Also, the pdf of such distribution cannot increase smoothly from zero at \( w \); there must be a jump. This rules a few more candidate distributions such as the lognormal or the Fréchet. Furthermore, the lowest possible value of the random variable cannot be an isolated atom, which rules out all discrete distributions such as the two-point distribution, the binomial, negative binomial, Poisson, etc.

Ultimately, it must be remembered that the aforementioned derivation of the Weibull distribution requires that the technologies consist of a large number of complementary components. If this were not the case, i.e., if technologies were rather simple (consisting of a few components only) or if their components were mutually substitutable, then the current Weibull result would not follow. One particularly instructive case of such a departure, discussed in more detail in the appendix, is the following. Namely, if one substituted the minimum of \( n \) draws from distribution \( F \) (with \( n \to \infty \)) with the maximum of these draws, then the limiting distribution would be Fréchet instead of Weibull (cf. Kortum, 1993; Kotz and Nadarajah, 2000; Jones, 2005). In consequence, one would obtain the same functional form of the technology menu (2), but with \( \alpha < 0 \); consequently, the resulting aggregate production function would still be (normalized) CES, but with gross substitutability between inputs \( (\sigma > 1) \) instead of gross complementarity. The limiting case where we take the maximum of \( n \) draws from distribution \( F \), with \( n \to \infty \) can be interpreted as the case where technologies are simple – consist of a single component – but the R&D is a repeated “trial-and-error” process where only the best try is considered useful. Hence, this case would correspond to a world where researchers work on simple, narrowly focused projects and are allowed to redo these projects over and over again in the search for improvements. In the light of this interpretation, the Fréchet case seems to be markedly less plausible empirically than the Weibull case, especially nowadays, in line with the empirical findings reviewed by Chirinko (2008) which indicate that the aggregate elasticity of substitution between capital and labor is below unity, both in the short and in the long run.
5.2 Derivation of the technology menu

Let us finally show how the individual draws of (complex, factor-augmenting) technologies $\tilde{a}$ and $\tilde{b}$ are combined, yielding the functional form of the technology menu postulated in Assumptions 2 and 4. We shall close the model of the two independent R&D sectors by making the following assumption.

**Assumption 6** Every capital- or labor-augmenting technology draw is allowed to enter the technology menu if it has been confirmed by at least a pre-defined fraction of researchers in $I$ ($z_b$ or $z_a$, respectively).

In the parlance of the above assumption, a “confirmed” technology is such that the given fraction of researchers has simultaneously obtained the same or a higher technology draw ($\tilde{b}$ or $\tilde{a}$, respectively).

Formally, given the above assumption and the Law of Large Numbers, a labor-augmenting technology $a$ will be included in the technology menu at time $t$ if and only if $P(\tilde{a} > a) \geq z_a$, and a capital-augmenting technology $b$, if and only if $P(\tilde{b} > b) \geq z_b$. Since both R&D sectors are independent from one another, it follows that a technology pair $(a, b)$ is included in the technology menu if $P(\tilde{a} > a, \tilde{b} > b) = P(\tilde{a} > a)P(\tilde{b} > b) \geq z_az_b$.\(^{24}\)

Furthermore, since no profit-maximizing firm would ever choose a dominated technology, we may as well replace the above “$\geq$” inequality with equality in the formulation of the technology menu. This brings us directly to Assumption 2, if the distributions of $\tilde{a}$ and $\tilde{b}$ are Weibull, or to Assumption 4, if these distributions are Pareto. In the light of the discussion above, if each idea consists of $n$ complementary components, then as $n \rightarrow \infty$, these distributions should rather converge to the Weibull, in line with Assumption 2.

6 Conclusion

The objective of the current paper has been to provide an idea-based microfoundation for the aggregate normalized CES production function. To this end, we have proposed an

\(^{24}\)Please note that the last inequality could also be assumed directly, leading to a generalization of our framework (see Appendix D in Growiec, 2008a). In such case, different pairs of $z_a$ and $z_b$ (such that their product $z_az_b$ is given) would be allowed at the technology menu simultaneously, whereas currently we fix the values of $z_a$ and $z_b$ separately. This extension does not bring about any significant change in results, however, and thus we leave it aside.
“endogeneous technology choice” model where this function is obtained as a convex hull of local production techniques, under the assumption that unit productivities of capital and labor ideas are independently Weibull-distributed (Growiec, 2008a,b). We have also demonstrated that if they are independently Pareto-distributed, then the resultant aggregate production function will be Cobb–Douglas (Jones, 2005), and if neither of these options holds, then the resultant aggregate production function will not belong to the CES class.

The discussed model has a number of interesting features. First, thanks to normalization, all parameters of the derived aggregate production function have a sound interpretation in terms of the parameters of local production functions and the underlying unit factor productivity (UFP) distributions. Second, we find that the elasticity of substitution between inputs along the aggregate production function is unambiguously higher than along the local production techniques, signifying that if the production function is viewed as an assembly of heterogeneous technologies, then technological substitution can effectively augment factor substitution. However, the elasticity of substitution along the aggregate production function is still bounded from above by unity, and thus capital and labor are gross complements. Third, normalization can be maintained simultaneously at the local and aggregate level.

The next step taken in the current paper was to embed our static “endogeneous technology choice” framework in a dynamic growth model. This step provided us with an opportunity to derive a number of theoretical predictions regarding the direction of technical change, endogeneously determined by the firms. Marked differences have been found here between the normalized CES case and the Cobb–Douglas case: the CES case allows for any direction of factor-augmenting technical change over the long run, and this direction is positively related to but typically not equal to the direction of R&D; the Cobb–Douglas case, on the other hand, implies that technical change must be purely labor-augmenting over the long run (along the balanced growth path), regardless of the underlying direction of R&D.

Perhaps the most important contribution of the current paper to the literature has been to develop a novel, tractable model of directed R&D, underlying the postulated UFP distributions. Using this model, we have provided a strong theoretical argument why the Weibull distribution should in fact be a good proxy of the real-world UFP distributions. The argument is based on the assumption that ideas (technologies, pro-
duction techniques) are not simple, as it was implicitly assumed in earlier literature, but inherently complex, consisting of a large number of complementary components. Under such circumstances, the efficiency of a given technology should be closely following the efficiency of its “weakest link”, i.e., the least efficient component. The Weibull distribution is, in turn, the extreme value distribution, characterizing the minimum of $n$ random draws from the same underlying distribution (which is bounded from below), in the limit of $n \to \infty$. Consequently, we have shown that if technologies consist of a wide range of complementary components, and they are then optimally chosen by firms, then the aggregate production function should be CES, rather than Cobb–Douglas.

In the appendix, we also demonstrate that all our arguments are readily generalizable to the case of $n$-input production functions. We also indicate how gross substitutability of inputs along the aggregate production function could be generated thanks to some specific changes in assumptions, and how to reinterpret some of our results in terms of technology adoption costs.

There is a wide range of issues, closely related to the current paper, that should be studied in further research. Let us just name a few. Firstly, it would be worthwhile to investigate the real-world productivity distributions underlying our setup, attempting to discriminate econometrically between the Weibull specification and the celebrated Pareto one (or perhaps some further distributions, too). Another challenge would be to develop an empirical approach able to identify jointly the parameters of the aggregate production function and the technology menu. Thirdly, it would be interesting to see the consequences of allowing for dependence between the marginal Weibull distributions. Fourthly, the current model could also be potentially applied in the modeling of multi-stage production processes, or used as means to motivate or endogeneize selected dimensions of firm heterogeneity.

References


Appendices

A Generalization to n inputs (and proofs of propositions)

As announced in the main text, all our results go through in the general case of n-input production functions as well. Let us now discuss this case.

A.1 The normalized CES case

First, let us show that if ideas (UFPs), augmenting each of the n production inputs, are independently Weibull-distributed (and the LPFs are normalized CES or Leontief functions), then the resultant aggregate production is normalized CES as well. To this end, we shall use the following generalized assumptions. By \( x_i, i = 1, 2, ..., n \) we shall denote the inputs, and by \( a_i, i = 1, 2, ..., n \) - unit factor productivities.

Assumption 7 The n-input local production function (LPF) takes either the normalized CES or the normalized Leontief form:

\[
Y = \begin{cases} 
Y_0 \left( \sum_{i=1}^{n} \pi_{0i} \left( \frac{a_i x_i}{a_0 x_0} \right)^\theta \right)^\frac{1}{\theta}, & \text{if } \sigma_{LPF} \in (0, 1), \\
Y_0 \min_{i=1,...,n} \left\{ \left( \frac{a_i x_i}{a_0 x_0} \right) \right\}, & \text{if } \sigma_{LPF} = 0,
\end{cases}
\]

(32)

where \( \theta \in [-\infty, 0) \) is the substitutability parameter, related to the elasticity of substitution along the LPF via \( \sigma_{LPF} = \frac{1}{1-\theta} \). Leontief LPFs, with \( \sigma_{LPF} = 0 \), are obtained as a special case of the more general normalized CES class of LPFs by taking the limit \( \theta \rightarrow -\infty \) (we denote this case as \( \theta = -\infty \) for simplicity). \( \pi_{0i} \) is the income share of i-th factor at \( t_0 \). Factor income shares sum up to unity:

\[
\sum_{i=1}^{n} \pi_{0i} = 1,
\]

(33)

and the LPF exhibits constant returns to scale.
Assumption 8 The technology menu, specified in the \((a_1, ..., a_n)\) space, is given by the equality:

\[
H(a_1, ..., a_n) = \sum_{i=1}^{n} \left( \frac{a_i}{\lambda_{ai}} \right)^{\alpha} = N, \quad \lambda_{a1}, ..., \lambda_{an}, \alpha, N > 0.
\] (34)

The technology menu is understood as a contour line of the cumulative distribution function of the joint \(n\)-variate distribution of factor-augmenting ideas \(\tilde{a}_i, i = 1, ..., n\). Under independence of the \(n\) dimensions (so that marginal distributions are multiplied by one another), equation (34) obtains if and only if the marginal distributions are Weibull with the same shape parameter \(\alpha > 0\) (Growiec, 2008b):

\[
P(\tilde{a}_i > a_i) = e^{-\left(\frac{a_i}{\lambda_{ai}}\right)^{\alpha}}, \quad i = 1, 2, ..., n,
\] (35)

where all \(a_i > 0\). Under such parametrization, we have

\[
P(\tilde{a}_1 > a_1, ..., \tilde{a}_n > a_n) = e^{-\sum_{i=1}^{n} \left(\frac{a_i}{\lambda_{ai}}\right)^{\alpha}},
\] (36)

and thus the parameter \(N\) in equation (34) is interpreted as \(N = -\ln P(\tilde{a}_1 > a_1, ..., \tilde{a}_n > a_n) > 0\).

The case where \(\tilde{a}_i, i = 1, ..., n\) are independently Pareto-distributed leads to a different specification of the technology menu and will be considered separately in the next subsection. If they are Weibull-distributed but dependent, or independent but following some other distribution than Pareto or Weibull, the resultant aggregate production does not belong to the CES class and will not be considered here.

Assumption 9 Firms choose the technology \(n\)-tuple \((a_1, ..., a_n)\) optimally, subject to the current technology menu, such that their profit is maximized:

\[
\max_{a,b} \left\{ Y_0 \left( \sum_{i=1}^{n} \frac{a_i x_i}{a_0 x_0} \right)^{\frac{\theta}{\sigma}} \right\} \quad \text{s.t.} \quad \sum_{i=1}^{n} \left( \frac{a_i}{\lambda_{ai}} \right)^{\alpha} = N.
\] (37)

Factor remuneration, taken into account in the firms’ profit maximization problem, does not depend on the chosen technology so it can be safely omitted from the above optimization problem.\(^{25}\)

Finally, second order conditions require us to assume that \(\alpha > \theta\), so that the interior stationary point of the above optimization problem is a maximum. For the resultant

\(^{25}\)In the case of Leontief LPFs, optimization requires \(\frac{a_i x_i}{a_0 x_0} = \frac{a_j x_j}{a_0 x_0}\) for all \(i, j = 1, ..., n\).
aggregate production function to be concave with respect to \( x_i, i = 1, ..., n \), we need to assume furthermore that \( \alpha - \theta - \alpha \theta > 0 \). All these conditions are satisfied automatically in the case \( \alpha > 0 > \theta \), on which we concentrate here. The inputs are gross complements along the aggregate production function.

Again, our framework provides direct results on the firm’s optimal technology choice. First, at time \( t_0 \), when \( Y = Y_0 \) and \( x_i = x_{0i}, \lambda_{ai} = \lambda_{a0i} \) is assumed for all \( i = 1, ..., n \), the optimal technology choice satisfies:

\[
a_{0i}^* = (N\pi_{0i})^{\frac{1}{\alpha}} \lambda_{a0i}, \quad i = 1, ..., n,
\]

where \( \lambda_{a0i} \) is the value of \( \lambda_{ai} \) at time \( t_0 \). Values of \( a_{0i}^* \) will be used as \( a_{0i} \) in the normalization at the local level in all subsequent derivations.

For any other moment in time \( t \neq t_0 \), the optimal technology choices are:

\[
\left( \frac{a_j}{a_{0j}} \right)^* = \frac{\lambda_{aj}}{\lambda_{a0j}} \left( \sum_{i=1}^{n} \pi_{0i} \left( \frac{\lambda_{ai}}{\lambda_{a0i}} \frac{x_{ai}x_{0j}}{x_{a0i}x_{0j}} \right)^{\frac{\alpha \theta}{\alpha - \theta}} \right)^{-\frac{1}{\alpha}},
\]

for all \( j = 1, ..., n \), where \( \frac{\alpha \theta}{\alpha - \theta} \) is substituted with \( -\alpha \) in the case of Leontief LPFs (\( \theta = -\infty \)).

Inserting these optimal technology choices into the LPF, we obtain the following aggregation result.

**Proposition 4** If Assumptions 7-9 hold, then the aggregate production function takes the normalized CES form:

\[
Y = Y_0 \left( \sum_{i=1}^{n} \pi_{0i} \left( \frac{\lambda_{ai}}{\lambda_{a0i}} \frac{x_{ai}}{x_{0i}} \right)^{\frac{\alpha \theta}{\alpha - \theta}} \right)^{\frac{\alpha - \theta}{\alpha \theta}}.
\]

Again, \( \frac{\alpha \theta}{\alpha - \theta} \) is substituted with \( -\alpha \) in the case of Leontief LPFs. Hence, the normalized CES result obtains both in the case of CES and Leontief LPFs.

**Proof.** The proof is straightforward and requires just algebraic manipulations. To prove Proposition 1, one should simply take \( n = 2 \) in the following calculations.

First, in the case of CES LPFs, we form the Lagrangean:

\[
\mathcal{L} = Y_0 \left( \sum_{i=1}^{n} \pi_{0i} \left( \frac{a_{ai}x_{ai}}{a_{0ai}x_{0i}} \right)^{\theta} \right)^{\frac{1}{\theta}} + \Lambda \cdot \left\{ \sum_{i=1}^{n} \left( \frac{a_{ai}}{\lambda_{ai}} \right)^{\alpha} - N \right\}.
\]
Differentiating it with respect to $a_i$, $i = 1, \ldots, n$, and getting rid of $\Lambda$ yields:

$$
\left( \frac{a_i}{a_j} \right)^{\alpha-\theta} = \frac{\pi_0 i}{\pi_0 j} \left( \frac{\lambda_{ai}}{\lambda_{aj}} \right)^{\alpha} \left( \frac{x_i a_0,j}{x_j a_0,i} \right)^{\theta},
$$

(42)

for all $i, j = 1, \ldots, n$. Considering first the reference point of time $t_0$, when $x_i = x_{0,i}$, $\lambda_{ai} = \lambda_{a0i}$, $a_i = a_{0i}$ for all $i = 1, \ldots, n$, we obtain:

$$
a_{0i} = \left( \frac{\pi_0 i}{\pi_0 j} \right)^{\frac{1}{\alpha}} \frac{\lambda_{a0i}}{\lambda_{a0j}}.
$$

(43)

Using the specification of the technology menu (34) as well as the assumption that $\sum_{i=1}^n \pi_0 i = 1$, we obtain:

$$
a_{0i}^* = (N \pi_0 i)^{\frac{1}{\alpha}} \lambda_{a0i}, \quad i = 1, \ldots, n.
$$

(44)

For $t \neq t_0$, by plugging (44) into (42), using (34) again and rearranging, we obtain that:

$$
\left( \frac{a_j}{a_{0j}} \right)^* = \frac{\lambda_{aj}}{\lambda_{a0j}} \left( \sum_{i=1}^n \pi_0 i \left( \frac{\lambda_{ai} \lambda_{a0j} x_{i,x_{0j}}}{\lambda_{aj} \lambda_{a0i} x_{0i,x_j}} \right)^{\frac{\alpha \theta}{\alpha-\theta}} \right)^{-\frac{1}{\alpha}},
$$

(45)

for all $j = 1, \ldots, n$.

Plugging this into the LPF (32) and rearranging, we obtain the final result.

Given our parametric assumptions, second-order conditions for the maximization of the Lagrangean hold. To demonstrate this, it is useful to note that maximizing $L$ is equivalent to minimizing the following transformed Lagrangean $L_{min}$ (where the maximand function is taken to the power $\theta < 0$ for simplicity):

$$
L_{min} = Y_0^\theta \sum_{i=1}^n \pi_0 i \left( \frac{a_i x_{i}}{a_{0i} x_{0i}} \right)^{\theta} + \Lambda_{min} \cdot \left\{ \sum_{i=1}^n \left( \frac{a_i}{\lambda_{ai}} \right)^{\alpha} - N \right\}.
$$

(46)

We obtain the following second-order derivatives of $L_{min}$ (after inserting the first order condition to get rid of $\Lambda_{min}$):

$$
\frac{\partial^2 L_{min}}{\partial a_i^2} = \theta (\theta - \alpha) Y_0^\theta \pi_0 i \left( \frac{a_i x_{i}}{a_{0i} x_{0i}} \right)^{\theta} \frac{1}{a_i^2} > 0,
$$

(47)

$$
\frac{\partial^2 L_{min}}{\partial a_i \partial a_j} = 0,
$$

(48)

and thus $L_{min}$ is minimized.
In the case of Leontief LPFs, instead of forming the Lagrangean, one should use the equality \( \frac{a_i x_i}{a_{0i} x_0} = \frac{a_j x_j}{a_{0j} x_0} \) for all \( i, j = 1, ..., n \) – which must hold because of the assumption that the representative firm maximizes profits. Since equations (34) and (44) still hold, plugging these equalities into the LPF yields

\[
Y = Y_0 \frac{a_1 x_1}{a_{01} x_{01}} = Y_0 \left( \sum_{i=1}^{n} \pi_{0i} \left( \frac{\lambda_{ai} x_i}{\lambda_{a0i} x_{0i}} \right)^{-\alpha} \right)^{-\frac{1}{\alpha}}. \tag{49}
\]

Please note that the same result is obtained by taking the case of CES LPFs and considering the limit \( \theta \to -\infty \).

The corollary on factor income shares goes through in the \( n \)-dimensional case as well:

**Corollary 3** Assuming that factors are priced at their marginal product, the factor income shares are equal to:

\[
\pi_i = \frac{\pi_{0i} \left( \frac{\lambda_{ai} x_i}{\lambda_{a0i} x_{0i}} \right)^{\frac{\alpha \theta}{\alpha - \theta}}}{\sum_{i=1}^{n} \pi_{0i} \left( \frac{\lambda_{ai} x_i}{\lambda_{a0i} x_{0i}} \right)^{\frac{\alpha \theta}{\alpha - \theta}}}, \quad i = 1, ..., n. \tag{50}
\]

\[
(51)
\]

**A.2 The Cobb–Douglas case**

Let us now replace Assumption 8 with the following one:

**Assumption 10 (modification of Assumption 8)** The technology menu, specified in the \((a_1, ..., a_n)\) space, is given by the equality:

\[
H(a_1, ..., a_n) = \prod_{i=1}^{n} \left( \frac{a_i}{\lambda_{ai}} \right)^{\phi_i} = N, \quad \phi_i > 0, i = 1, ..., n. \tag{52}
\]

The current shape of the technology menu is consistent with the assumption that \( \tilde{a}_i \)’s are independently Pareto-distributed, with respective shape parameters \( \phi_i \). In such case, \( N = \frac{1}{P(\tilde{a}_1 > a_1, ..., \tilde{a}_n > a_n)} > 1 \).

At \( t_0 \), when \( Y = Y_0 \) and \( x_i = x_{0i}, \lambda_{ai} = \lambda_{a0i} \) is assumed for \( i = 1, ..., n \), the optimal choice is indeterminate, provided that

\[
\pi_{0i} = \frac{\phi_i}{\sum_{i=1}^{n} \phi_i}, \quad i = 1, ..., n. \tag{53}
\]
This restriction means that the factor income shares should be equal to \( \frac{\phi_i}{\sum_{i=1}^{n} \phi_i} \). Thus, \( \pi_{02}, ..., \pi_{0n} \) cease to be free parameters, and \( a_{02}, ..., a_{0n} \) become free parameters instead (the remaining technology choice \( a_{01} \) is then calculated according to the technology menu).

At any other moment in time \( t \neq t_0 \), and given \( a_{0i}, i = 1, ..., n \), the optimal technology choices are:

\[
\left( \frac{a_i}{a_{0i}} \right)^* = \frac{x_{0i}}{x_i} \prod_{i=1}^{n} \left( \frac{\lambda_{ai}}{\lambda_{a0i}} \frac{x_i}{x_{0i}} \right)^{\frac{\phi_i}{\sum_{i=1}^{n} \phi_i}}, \quad i = 1, ..., n.
\]

Inserting these optimal technology choices into the LPF, we obtain the following result.

**Proposition 5** If Assumptions 7, 9, and 10 hold, then the aggregate production function takes the normalized Cobb–Douglas form:

\[
Y = Y_0 \prod_{i=1}^{n} \left( \frac{\lambda_{ai}}{\lambda_{a0i}} \frac{x_i}{x_{0i}} \right)^{\frac{\phi_i}{\sum_{i=1}^{n} \phi_i}}.
\]

**Proof.** The proof is, again, straightforward and requires just algebraic manipulations. To prove Proposition 2, one should simply take \( n = 2 \) in the following calculations.

First, we form the Lagrangean:

\[
\mathcal{L} = Y_0 \left( \sum_{i=1}^{n} \pi_{0i} \left( \frac{a_i x_i}{a_{0i} x_{0i}} \right)^\theta \right)^{\frac{1}{\theta}} + \Lambda \cdot \left\{ \prod_{i=1}^{n} \left( \frac{a_i}{\lambda_{ai}} \right)^{\phi_i} - N \right\}.
\]

Differentiating it with respect to \( a_i, i = 1, ..., n \), and getting rid of \( \Lambda \) yields:

\[
\left( \frac{a_i x_i a_{0j} x_{0j}}{a_j x_j a_{0i} x_{0i}} \right)^\theta \frac{\pi_{0i} \phi_j}{\pi_{0j} \phi_i} = 1,
\]

for all \( i, j = 1, ..., n \). Considering first the reference point of time \( t_0 \), when \( x_i = x_{0i}, \lambda_{ai} = \lambda_{a0i}, a_i = a_{0i} \) for all \( i = 1, ..., n \), we obtain:

\[
\frac{\pi_{0i}}{\pi_{0j}} = \frac{\phi_i}{\phi_j}.
\]

Using the assumption that \( \sum_{i=1}^{n} \pi_{0i} = 1 \), we obtain that at \( t_0 \), optimal technology choice is indeterminate provided that:

\[
\pi_{0i} = \frac{\phi_i}{\sum_{i=1}^{n} \phi_i}, \quad i = 1, ..., n.
\]
For $t \neq t_0$, by plugging (58) into (57), using (52) and rearranging, we obtain that:
\[
\left( \frac{a_i}{a_{0i}} \right)^* = \frac{x_{0i}}{x_i} \prod_{i=1}^{n} \left( \frac{\lambda_{ai}}{\lambda_{a0i} x_{0i}} \right)^{\frac{\phi_i}{\sum_{i=1}^{n} \phi_i}} , \quad i = 1, ..., n.
\] (60)
for all $j = 1, ..., n$.

Plugging this into the LPF (32) and rearranging, we obtain the final result.

Given our parametric assumptions, second-order conditions for the maximization of the Lagrangean hold. To prove this, it is useful to note that maximizing $L$ is equivalent to minimizing the following transformed Lagrangean $L_{\text{min}}$ (where, for simplicity, the maximand function is taken to the power $\theta < 0$ and a log-transformation is applied to the restriction):
\[
L_{\text{min}} = Y^\theta \prod_{i=1}^{n} \left( \frac{a_i x_i}{a_{0i} x_{0i}} \right)^{\theta} + \sum_{i=1}^{n} \phi_i (\ln a_i - \ln \lambda_{ai}) - \ln N.
\] (61)

We obtain the following second-order derivatives of $L_{\text{min}}$ (after inserting the first order condition to get rid of $\Lambda_{\text{min}}$):
\[
\frac{\partial^2 L_{\text{min}}}{\partial a_i^2} = \theta^2 Y^\theta \prod_{i=1}^{n} \left( \frac{a_i x_i}{a_{0i} x_{0i}} \right)^{\theta} \frac{1}{a_i^2} > 0,
\] (62)
\[
\frac{\partial^2 L_{\text{min}}}{\partial a_i \partial a_j} = 0,
\] (63)
and thus $L_{\text{min}}$ is minimized.

In the case of Leontief LPFs, instead of forming the Lagrangean, one should use the equality $\frac{a_i x_i}{a_{0i} x_{0i}} = \frac{a_j x_j}{a_{0j} x_{0j}}$ for all $i, j = 1, ..., n$ – which must hold because of the assumption that the representative firm maximizes profits. Since equation (52) still holds, plugging these equalities into the LPF yields
\[
Y = Y_0 \frac{a_1 x_1}{a_{01} x_{01}} = Y_0 \prod_{i=1}^{n} \left( \frac{\lambda_{ai}}{\lambda_{a0i} x_{0i}} \right)^{\frac{\phi_i}{\sum_{i=1}^{n} \phi_i}}.
\] (64)

**B Aggregate CES production function with gross substitutability**

As spelled out in the main text, our “endogeneous technology choice” framework implies that if:
• LPFs are Leontief or CES with $\theta < 0$, so that inputs are gross complements along the LPF,

• the technology menu is generated as a contour line of a bivariate distribution whose marginals are independent Weibull distributions (and thus $\alpha > 0$),

then the aggregate production function is CES with an elasticity of substitution bounded from above by unity ($\sigma < 1$). Hence, capital and labor are necessarily gross complements not only along the LPF, but also along the aggregate production function. However, the Cobb–Douglas limit of $\sigma \to 1$ is approached when $\theta \to 0$ (so that the LPFs tend to Cobb–Douglas functions themselves) or when $\alpha \to 0$ (so that the Weibull distributions of UFPs get ever fatter tails).

As quoted in the main text, the implication that $\sigma < 1$ is in good agreement with empirical evidence surveyed by Chirinko (2008) and León-Ledesma et al. (2010). Despite that, it should be mentioned that our theoretical procedure can also be easily generalized to accommodate the case with $\sigma > 1$ as well. This could be done in two alternative ways, discussed in the following subsections. Both of them are analytically straightforward.

Furthermore, when dealing with the case of gross substitutability of inputs along the aggregate production function, one must remember that in such case, factor income shares move in the opposite direction than in the case of gross complementarity. The direction of their change is still strictly determined by the growth rate of $\lambda_b K$ relative to $\lambda_a L$, and if both growth rates are equal, then the capital income share will still remain constant at $\pi_0$. If $\lambda_b K$ grows faster, however, then in the gross substitutability case, the capital income share will gradually increase to unity over time; conversely, it will gradually fall to zero if $\lambda_a L$ grows faster.

Another caveat is that gross substitutability of inputs along the aggregate production function dramatically changes the conclusions regarding the long-run direction of technical change, due to two reasons: first, gross substitutability acts in favor of increasing the UFP of the relatively abundant factor instead of the relatively scarce factor. Since capital is the accumulable factor here, this implies that technical change should be largely capital-augmenting, rather than labor-augmenting as in the case of gross complementarity, especially in the long run (cf. Klump and de La Grandville, 2000; Acemoglu,

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26In Growiec (2008a), where the UFP distributions were modeled as mutually dependent Pareto distributions, the possibility of $\sigma > 1$ was allowed, too.
2003; Growiec, 2008a). Second, gross substitutability can act as an engine of perpetual growth due to capital accumulation alone (de La Grandville, 1989; Klump and Preissler, 2000; Palivos and Karagiannis, 2010). A precise discussion of these issues is beyond the scope of the current paper.

B.1 The case with $\theta \geq 0$

The first of the two possibilities allowing for gross substitutability of inputs along the aggregate production function is to assume that inputs are gross substitutes already along the LPF (so that $\theta > 0$), or at least that the LPFs are Cobb–Douglas (which happens the limiting case $\theta \to 0$).

It is easily verified that if $\theta \in (0, \alpha)$ then all our derivations still go through, and Proposition 1 still holds, but this time with $\theta > 0$. The aforementioned restriction $\alpha > \theta$ is required for the second-order optimality conditions to hold (see the proof of Proposition 1 in Appendix A). On top of that, we also need to assume that $\alpha - \theta - \alpha \theta \geq 0$ (or $\theta \leq \frac{\alpha}{1+\alpha}$) to ensure that the aggregate production function is concave with respect to its arguments:

$$\sigma = \frac{\alpha - \theta}{\alpha - \theta - \alpha \theta} \in (1, +\infty). \tag{65}$$

Please note that as $\theta \to \left(\frac{\alpha}{1+\alpha}\right)^-$ then the elasticity of substitution of the aggregate production function tends to infinity, and the function becomes linear in the limit. Conversely, when $\theta \to 0^+$ then $\sigma \to 1^+$ and the aggregate production function converges to the Cobb–Douglas form.

This particular functional form may also be derived directly if LPFs are assumed to be Cobb–Douglas themselves (cf. Growiec, 2008a). To this end, let us assume that the LPFs take the form

$$Y = Y_0 \left( bK_{b0}K_0 \right)^\psi \left( aL_{a0}L_0 \right)^{1-\psi}, \tag{66}$$

with $\psi = \pi_0$ being the initial capital income share, which will soon turn out to be constant for all $t$. Indeed, it is now quickly verified that at any moment in time $t$, the optimal technology choice satisfies:

$$a^* = \lambda_a \left( N(1-\psi) \right)^{\frac{1}{\alpha}}, \quad b^* = \lambda_b \left( N\psi \right)^{\frac{1}{\alpha}}, \tag{67}$$

and thus

$$\left( \frac{a}{a_0} \right)^* = \frac{\lambda_a}{\lambda_{a0}}, \quad \left( \frac{b}{b_0} \right)^* = \frac{\lambda_b}{\lambda_{b0}}, \tag{68}$$
implying that the aggregate production function takes the Cobb–Douglas form with the same partial elasticity of capital $\psi$ as the LPF:

$$Y = Y_0 \left( \frac{\lambda_b K}{\lambda_{b0} K_0} \right)^\psi \left( \frac{\lambda_c L}{\lambda_{c0} L_0} \right)^{1-\psi}. \tag{69}$$

Despite the analytical simplicity of the above derivations, it must be remembered that the assumption that inputs are gross substitutes along the LPF is widely at odds with the “recipe” interpretation of the LPF, which views it as a set of strictly specified instructions, indicating how to turn inputs into output (Jones, 2005). Given this interpretation, one should rather expect the LPFs to be approximately Leontief, than to have more than unitary elasticity of substitution.

### B.2 The case with $\alpha < 0$

The second of the two possibilities allowing for gross substitutability of inputs along the aggregate production function is to assume that the parameter $\alpha$ in the specification of the technology menu (2) is negative.

If $\alpha \in (\theta, 0)$ then all our derivations still go through, and Proposition 1 still holds, but this time with $\alpha < 0$. The restriction $\alpha > \theta$ is required for the second-order optimality conditions to hold (see the proof of Proposition 1 in Appendix A). We need to assume furthermore that $\alpha - \theta - \alpha \theta \geq 0$ (or $\alpha \geq \frac{\theta}{1-\theta}$) to ensure that the aggregate production function is concave with respect to its arguments:

$$\sigma = \frac{\alpha - \theta}{\alpha - \theta - \alpha \theta} \in (1, +\infty). \tag{70}$$

Please note that as $\alpha \to \left( \frac{\theta}{1-\theta} \right)_+$ then the elasticity of substitution of the aggregate production function tends to infinity, and the function becomes linear in the limit. Conversely, when $\alpha \to 0_-$ then $\sigma \to 1_+$ and the aggregate production function converges to the Cobb–Douglas form, just like it does when $\alpha \to 0_+$. Even if $\alpha < 0$, the technology menu (2) can still be derived as a contour line of the cumulative distribution function of the joint bivariate distribution of capital-augmenting ideas $\tilde{b}$ and labor-augmenting ideas $\tilde{a}$. We find that under independence of both dimensions (so that marginal distributions are multiplied by one another), equation (2) with $\alpha < 0$ obtains if and only if the marginal distributions are Fréchet with the same shape parameter $\alpha < 0$:

$$P(\tilde{a} \leq a) = e^{-\left(\frac{a}{\tilde{a}}\right)^\alpha}, \quad P(\tilde{b} \leq b) = e^{-\left(\frac{b}{\tilde{b}}\right)^\alpha}, \tag{71}$$

$$40$$
for $a, b > 0$. Under such parametrization, we have $P(\bar{a} \leq a, \bar{b} \leq b) = e^{-\left(\frac{\bar{a}}{\lambda_a}\right)^\alpha - \left(\frac{\bar{b}}{\lambda_b}\right)^\alpha}$, and thus the parameter $N$ in eq. (2) is interpreted as $N = -\ln P(\bar{a} \leq a, \bar{b} \leq b) > 0$. As in the main text, we may assume $N$ to be constant across time, and $\lambda_a, \lambda_b$ to increase over time as an outcome of factor-augmenting R&D.

The Fréchet distribution is also an extreme value distribution, just like the Weibull distribution is (cf. de Haan and Ferreira, 2006); however, while the Weibull is min–stable, the Fréchet is max–stable. It means that it is the limiting distribution of the maximum of $n$ independent random draws from a given distribution $F$ bounded from below, approached when $n \to \infty$. Once $F$ happens to satisfy a number of technical conditions alike the ones required by Assumption 5, then the maximum of these draws, after appropriate normalization, will converge in distribution to the Fréchet distribution (for mathematical reference, see Kotz and Nadarajah, 2000; for economic applications, see Kortum, 1993; Jones, 2005).

Hence, the current change in assumptions has a profound impact on the interpretation of the R&D process. Namely, it requires one to assume that, instead of technologies being complex and consisting of a range of mutually complementary components, that technologies are simple – consist of a single component – but then the R&D is a repeated “trial-and-error” process where only the best try is considered useful. Hence, this case would correspond to a world where researchers work on simple, narrowly focused projects and are allowed to redo their projects over and over again in the search for improvements. In reality, however, technologies are becoming more and more sophisticated nowadays, in contrast to this stylized framework. This implies that the current case is less likely to be empirically plausible than the baseline case discussed in the main text.

C A reinterpretation of the Cobb–Douglas case in terms of technology adoption costs

As apparent from the recent contribution of León-Ledesma and Satchi (2011), the variant of the current “endogeneous technology choice” model that leads to the Cobb–Douglas result (cf. Section 4), could also be reinterpreted in terms of (Hicks-neutral) technology adoption costs. Namely, as posited by these authors, we could assume that the LPF
takes the following specific unnormalized CES form (see also Growiec, 2008a):

\[ Y = \Gamma f(\eta) \left( \eta (\lambda b K)^{\theta} + (1 - \eta) (\lambda a L)^{\theta} \right)^{\frac{1}{\theta}}, \quad \theta < 0, \Gamma > 0, \eta \in [0, 1], \]  

(72)

where the assumption \( \theta < 0 \) mirrors gross complementarity of inputs along the LPF.

The inclusion of the Hicks-neutral \( f(\eta) \) term in the local production function is meant to capture adoption costs in the production process. As reflected by the posited inverse U-shape of \( f(\eta) \), this specification implies that highly labor- or capital-intensive technologies are costly, whereas “intermediate” technologies, using both factors in moderation, are relatively cheap. All technology adoption costs are borne in the form of Hicks-neutral technical inefficiency.

Furthermore, assuming unit invariance of the LPF – that is, requiring that the functional form \( f(\eta) \) does not change with the units of labor measurement (e.g., hours worked, number of full-time equivalent employees, etc.) – León-Ledesma and Satchi (2011) obtain that

\[ f(\eta) = (\eta^{\gamma}(1 - \eta)^{1-\gamma})^{-\frac{1}{\gamma}}. \]  

(73)

Upon maximization of eq. (72) with respect to \( \eta \in [0, 1] \), it is obtained that the optimal choice of \( \eta \) satisfies:

\[ \frac{1 - \eta}{\eta} = \frac{1 - \gamma}{\gamma} \left( \frac{\lambda b K}{\lambda a L} \right)^{\theta}. \]  

(74)

Inserting this optimal choice into (72), the aggregate production function is derived as:

\[ Y = \Gamma \left( \gamma^{\gamma}(1 - \gamma)^{1-\gamma} \right)^{-\frac{1}{\gamma}} \left( \lambda b K \right)^{\gamma} \left( \lambda a L \right)^{1-\gamma}. \]  

(75)

Hence, the aggregate production function is Cobb–Douglas, just like in Proposition 2. In consequence, it follows that the assumption of Hicks-neutral technology adoption costs, coupled with unit invariance, is equivalent to assuming that the technology menu takes the form (21), consistent with the assumption that UFPs are independently Pareto-distributed.

Two caveats remain when discussing this analogy, though. First, the adoption cost mechanism proposed by León-Ledesma and Satchi (2011), despite its analytical simplicity and intuitive appeal, does not take normalization of CES functions into account. Under normalization, however, the CES production function is itself invariant to the choice of units of measurement of capital and labor, and thus imposing unit invariance on top of that does not place any further restrictions on the functional form of \( f(\eta) \), rendering the analytical assumption (73) unmotivated. This implication could potentially
generate a wider variety of aggregate production functions that could be derived using
the current “adoption costs” framework once the restriction (73) is relaxed. Second,
the assumption that technology adoption costs are borne in the form of Hicks-neutral
technical inefficiency is very likely to play a role in generating the multiplicative (Cobb–
Douglas) form of the aggregate production function, too. If these costs were borne, for
instance, in the form of factor-specific UFP losses, then the Cobb–Douglas result would
likely fail, just as it fails in our current setup if the technology menu takes a different
form than the one required by Assumption 4.